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701. (a) Since $\cos x = 0$ has roots, this is not true. $x = \frac{\pi}{2} \neq k$ is a counterexample.
(b) Since $2^x = 0$ has no roots, this is true. If the product of $(x-k)$ and 2^x is zero, then it is $(x-k)$ that must be zero.

702. For a population with distribution $X \sim N(\mu, \sigma^2)$, the means of samples of size n are distributed as

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

So, the population mean is 50, and the population variance is given by $\frac{\sigma^2}{20} = 1.25$. Hence, $\sigma^2 = 25$.

703. (a) Differentiating, $f'(x) = -8x^{-3} + 3x^2 + 4$. This gives $f'(2) = -1+12+4 = 15$. Also, $f(2) = 24$.
(b) The best linear approximation to a curve is its tangent line. Passing through the point $(2, 24)$ with gradient 15, this is $y - 24 = 15(x-2)$. The best linear approximation is $g(x) = 15x - 6$.

704. This is a quadratic in $\cos x$. Factorising,

$$\begin{aligned} &\cos x(\cos x - 1) = 0 \\ \Longrightarrow\ &\cos x = 0, 1 \\ \Longrightarrow\ &x = -90°, 0°, 90°. \end{aligned}$$

705. The student has ignored the fact that a parabola can be stretched in the y direction while keeping the same vertex. So, the text should read: "If the vertex of a positive parabola is at (a, b), then the parabola must have equation $y = k(x-a)^2 + b$, where $k > 0$."

706. The average speed is given by

$$\bar{v} = \tfrac{1}{5}\Big[t^3 - t^2\Big]_0^5 = 20.$$

For instantaneous speed, we differentiate. This gives $v = 3t^2 - 2t$. Setting this equal to 20,

$$\begin{aligned} &3t^2 - 2t - 20 = 0 \\ \Longrightarrow\ &t = \frac{2 \pm \sqrt{4 - 4\cdot 3 \cdot -20}}{2 \cdot 3} \\ &\quad = \tfrac{1}{3}\big(1 \pm \sqrt{61}\big). \end{aligned}$$

The positive root satisfies $0 < \frac{1}{3}\big(1+\sqrt{61}\big) < 5$. So, at $t = \frac{1}{3}\big(1+\sqrt{61}\big)$, the instantaneous speed is the same as the average speed over the five seconds.

707. Substituting $x = 2y - \frac{7}{2}$ gives

$$\begin{aligned} &4\big(2y - \tfrac{7}{2}\big)^2 + y^2 = 5 \\ \Longrightarrow\ &17y^2 - 56y + 44 = 0 \\ \Longrightarrow\ &(17y - 22)(y - 2) = 0 \\ \Longrightarrow\ &y = 2, \tfrac{22}{17}. \end{aligned}$$

This gives the simultaneous solution points (x, y) as $\big(\frac{1}{2}, 2\big)$ and $\big(-\frac{31}{34}, \frac{22}{17}\big)$.

— Nota Bene —

Finding the factorisation in line three, you can feel free to *use* a calculator, even though the calculator can't give you the "determine" reasoning asked for in the question. Just solve the quadratic using the formula or a polynomial solver, and then reverse engineer the brackets using the factor theorem.

708. Counterexample: $u_5 = 5^2 + 2 = 3^3$.

709. The first sentence is

$$\int f(x) - g(x)\, dx = ax + b,$$

for some constants a, b. Differentiating both sides, we get $f(x) - g(x) = a$. So, $f(x)$ and $g(x)$ differ by a constant. If $f(x) = g(x)$ for any x, then this constant must be zero, and therefore $f(x) = g(x)$ for all x. □

710. The total of the interior angles of a pentagon is 3π, so the average is $\frac{3\pi}{5}$. The average of the smallest $\frac{\pi}{4}$ and largest θ must therefore also be $\frac{3\pi}{5}$. So,

$$\begin{aligned} &\tfrac{1}{2}\big(\tfrac{\pi}{4} + \theta\big) = \tfrac{3\pi}{5} \\ \Longrightarrow\ &\theta = \tfrac{19\pi}{20} \text{ radians.} \end{aligned}$$

711. (a) Substituting gives $\big(a^1 - a^0\big) - \big(a^0 - a^{-1}\big) = 0$. Since $a^0 = 1$, this is the required result.
(b) Multiplying up by a to deal with the negative power, $a^2 - 2a + 1 = 0 \implies a = 1$.

712. (a) Subtracting three right-angled triangles from the square, the area of the shaded region is

$$\begin{aligned} A &= 1 - \tfrac{1}{2}k - \tfrac{1}{2}k - \tfrac{1}{2}(1-k)^2 \\ &\equiv \tfrac{1}{2}\big(1 - k^2\big). \end{aligned}$$

(b) Differentiating gives $\frac{dA}{dk} = -k$.
(c) Setting $\frac{dA}{dk}$ for optimisation, we get $-k = 0$. So, the area is maximal when $k = 0$. This gives $A = \frac{1}{2}$.

— Alternative Method —

Since $k^2 \geq 0$, we know that $A = \frac{1}{2}\big(1-k^2\big) \leq \frac{1}{2}$. Setting $k = 0$ puts the vertices of the triangle at $(1, 0)$ and $(0, 1)$, giving area $A = \frac{1}{2}$. This area must therefore be maximal.

713. Completing the square gives $f(x) = 4(x+1)^2 - 2$ and $g(x) = (x+3)^2 - 2$. Each is a positive quadratic function with minimum output -2. Hence, they have the same range $[-2, \infty)$, as required.

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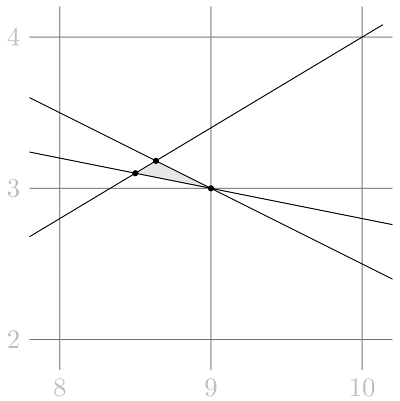
714. The boundary equations are

$$\begin{aligned} 3x - 5y &= 10, \\ x + 2y &= 15, \\ x + 5y &= 24. \end{aligned}$$

Solving these pairwise, the intersections are at $\left(\frac{95}{11}, \frac{35}{11}\right)$, $\left(\frac{17}{2}, \frac{31}{10}\right)$ and $(9, 3)$. All of these satisfy $8 < x \leq 9$ and $3 \leq y < 4$, i.e. they lie in the same grid square. Shading the relevant region on an integer grid:



So, the only integer point which satisfies all three inequalities simultaneously is $(9, 3)$.

715. (a) Using the iterative definition,

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 + 1 = 2 \\ u_3 &= 2 + 2 = 4 \\ u_4 &= 4 + 3 = 7 \\ u_5 &= 7 + 4 = 11. \end{aligned}$$

(b) Using u_1, we have $1 = k \cdot 1 \cdot 2$. So, $k = \frac{1}{2}$.

716. The result is true. Considered graphically, it is straightforward: the signed area from a to b plus the signed area from b to c is the signed area from a to c.

— Alternative Method —

Defining the indefinite integral F by $\mathrm{F}'(x) = \mathrm{f}(x)$, the LHS is $\mathrm{F}(b) - \mathrm{F}(a) + \mathrm{F}(c) - \mathrm{F}(b)$, which simplifies to $\mathrm{F}(c) - \mathrm{F}(a)$. This is the RHS.

717. Rearranging, $t = \frac{v-u}{a}$. Substituting gives

$$\begin{aligned} & s = u\left(\frac{v-u}{a}\right) + \tfrac{1}{2}a\left(\frac{v-u}{a}\right)^2 \\ \implies & as = u(v-u) + \tfrac{1}{2}(v^2 - 2uv + u^2) \\ \implies & as = uv - u^2 + \tfrac{1}{2}v^2 - uv + \tfrac{1}{2}u^2 \\ \implies & 2as = v^2 - u^2 \\ \implies & v^2 = u^2 + 2as, \text{ as required.} \end{aligned}$$

718. (a) $(-\infty, 2] \cup (-4, 6] = (-\infty, 6]$,
(b) $[0, \infty) \setminus (1, \infty) = [0, 1]$,
(c) $(-\infty, 1] \cap [-1, \infty) = [-1, 1]$.

719. The line $x = k$ is parallel to the y axis. So, if it is to be a normal, the tangent must be parallel to the x axis. Hence, set $\frac{dy}{dx} = 4x^3 - 64x = 0$. Solving this yields $x = 0, \pm 4$. So, $k = 0, \pm 4$.

720. Let $x = 0.\dot{2}3\dot{7}$. Multiplying by a thousand shifts the decimal point by three, so $1000x = 237.\dot{2}3\dot{7}$. Subtracting the two equations gives $999x = 237$. Hence, $x = \frac{237}{999}$, as required.

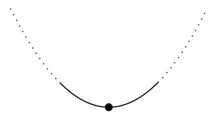
721. Defining x as positive upwards from ground level, the motion is from $x = h$ to $x = 0$. The weight is downwards, so has signed value $F = -mg$. The work done by gravity is therefore

$$\int_h^0 -mg\, dx = \Big[-mgx\Big]_h^0 = (0) - (-mgh) = mgh.$$

— Nota Bene —

The quantity mgh, which has units of Joules, is the *gravitational potential energy* close to the Earth's surface. Together with the *kinetic energy* $\frac{1}{2}mv^2$ and the principle of the conservation of energy, it can be a useful tool in problem-solving.

722. Since the first derivative is zero at $x = 2$, the point $(2, 3)$ is a stationary point. Furthermore, since the second derivative is positive, $(2, 3)$ is a minimum: the gradient is increasing. Hence, in the vicinity of $x = 2$, the graph looks approximately like a positive parabola:



— Nota Bene —

Depending on the value of the third derivative, the graph $y = \mathrm{f}(x)$ may depart from this behaviour very close to $x = 2$. However, f is a polynomial function, so cannot have any strange behaviours: if we zoom in close enough at $x = 2$, the graph must look approximately parabolic.

723. A4 is designed so that the area scale factor from A4 to the similar A5 is $2 : 1$; so the length scale factor is $\sqrt{2} : 1$.

The long sides of the rectangle are the diagonals of square faces, i.e. they are the hypotenuses of $\left(l, l, l\sqrt{2}\right)$ triangles. So, the rectangle has sides in the ratio $l\sqrt{2} : l$, which is $\sqrt{2} : 1$. These are the dimensions of A4, as required.

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724. (a) This statement holds. If the rationals x and y are quotients of integers $\frac{a}{b}$ and $\frac{c}{d}$, then their product $\frac{ac}{bd}$ is too.

(b) This statement doesn't hold. The set $\mathbb{R} \setminus \mathbb{Q}$ is the set of reals minus the set of rationals, i.e. it is the set of irrationals. A pair of surds such as $x = y = \sqrt{2}$ is a counterexample. These are irrational, so are elements of $\mathbb{R} \setminus \mathbb{Q}$. But their product is 2, which is rational.

725. (a) Expanding using the binomial formula,

$$\begin{aligned}(x+h)^n = \;& x^n + {}^n\mathrm{C}_1 x^{n-1}h \\ & + \text{ terms in } h^2 \text{ and higher}\end{aligned}$$

The binomial coefficient ${}^n\mathrm{C}_1$ simplifies as

$${}^n\mathrm{C}_1 = \frac{n!}{(n-1)!1!} = n.$$

This gives the required result:

$$\begin{aligned}(x+h)^n = \;& x^n + nx^{n-1}h \\ & + \text{ terms in } h^2 \text{ and higher}\end{aligned}$$

(b) We set up the standard first-principles limit and use the result of part (a) to simplify the numerator:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \text{ terms in } h^2 \text{ and higher}}{h} \\ &= \lim_{h\to 0} \big(nx^{n-1} + \text{ terms in } h \text{ and higher}\big) \\ &= nx^{n-1}, \text{ as required.}\end{aligned}$$

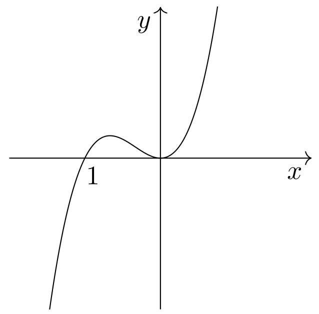
726. The possibility space is a 6×6 grid of 36 equally likely outcomes. The sum events are diagonals. 7 is the leading diagonal, which means it is the most likely sum. 6 and 8 are the next most likely.

	1	2	3	4	5	6
1					6	
2				6		8
3			6		8	
4		6		8		
5	6		8			
6		8				

So, the probabilities are equal.

727. (a) Differentiating, $h'(x) = 3x^2 + 2ax + b$ and $h''(x) = 6x + 2a$. Since $h''(0) = 2$, we know that $a = 1$. Then, using $h'(0) = \mathrm{h}(0) = 0$, we get $b, c = 0$. So, the function is $\mathrm{h}(x) = x^3 + x^2$.

(b) First, we factorise the cubic: $y = x^2(x+1)$. This has a double root at $x = 0$ and a single root at $x = -1$. Hence, at $x = 0$ the curve looks like a parabola, and at $x = -1$ it looks like a straight line.



728. (a) We know that the forces are equal due to NII. Since cow A is not accelerating, the resultant horizontal force on it must be zero. The only horizontal forces on cow A are friction from the ground and the contact force from cow B. Hence, the forces are equal in magnitude.

(b) We know that these forces are equal due to NIII. They are the two aspects of the single interaction between cow A and cow B, and are equal by definition. Unlike the forces in part (a), these forces would be equal whatever the acceleration of the cows.

729. The denominator has a root at $x = -\frac{1}{2}$, so this is the equation of the vertical asymptote. To find the horizontal asymptote, we divide the numerator and denominator by x, which gives

$$y = \frac{8 + {}^3\!/\!_x}{2 + {}^1\!/\!_x}.$$

As $x \to \pm\infty$, the inlaid fractions tend to zero, which means $y \to \frac{8}{2} = 4$. Hence, the horizontal asymptote is at $y = 4$.

730. There is a common factor of $\sqrt{2}$. Taking it out, we have $\sqrt{2}(x^2 + 2x + 4)$. Completing the square inside the bracket gives $\sqrt{2}\big((x+1)^2 + 3\big)$. We can then expand:

$$\sqrt{2}x^2 + \sqrt{8}x + \sqrt{32} \equiv \sqrt{2}(x+1)^2 + 3\sqrt{2}.$$

731. (a) Since the mean value of x_i is 10, the mean value of $x_i - 10$ is 0.

(b) For a sample element x_i, the quantity $(x_i - 10)^2$ is the squared deviation from the mean (in this case 10). The mean of this is the variance, by definition:

$$s^2 = \frac{\sum (x_i - 10)^2}{n}.$$

So, the required value is 25.

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732. Differentiation is a linear operation, which tells us that
$$\frac{d}{dx}(a+b) = \frac{d}{dx}(a) + \frac{d}{dx}(b).$$
So, we can apply the differential operator $\frac{d}{dx}$ term by term. The derivatives are
$$\frac{d}{dx}(2x) = 2, \quad \frac{d}{dx}(3y) = 3\frac{dy}{dx}, \quad \frac{d}{dx}(5) = 0.$$
Putting the above together,
$$\frac{d}{dx}(2x+3y+5) = 2 + 3\frac{dy}{dx}.$$

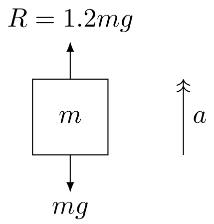
733. The first two functions are $x \mapsto x$ and $x \mapsto 1$, both of which may take any real input. The reciprocal function, however, cannot take the input zero:

(a) $\mathbb{R}$,

(b) $\mathbb{R}$,

(c) $\mathbb{R} \setminus \{0\}$.

734. Weighing scales do not, in fact, measure weight, but rather the reaction force at contact. For an object of weight mg, this is overestimated by 20%, so the upwards reaction force exerted by the scales on the object is $R = 1.2mg$.



The equation of motion is $1.2mg - mg = ma$, which gives an acceleration of $0.2g$ ms^{-2} upwards.

735. (a) This is true. Integration, like differentiation, is a linear operation, which means that, for constants a, b
$$\int a\,\mathrm{f}(x) + b\,\mathrm{g}(x)\,dx \equiv a\int \mathrm{f}(x)\,dx + b\int \mathrm{g}(x)dx.$$

(b) This is false. The integral can be split up term by term, but not as shown. The first term k would integrate to give $k(b-a)$:
$$\begin{aligned}&\int_a^b k + \mathrm{f}(x)\,dx\\ \equiv\ &\Big[kx\Big]_a^b + \int_a^b \mathrm{f}(x)\,dx\\ \equiv\ &k(b-a) + \int_a^b \mathrm{f}(x)\,dx.\end{aligned}$$

736. The curve is a circle. Completing the square, its equation is $(x+2)^2 + (y-1)^2 = 5$. A normal to a circle is a radius, which is from $(-2,1)$ to $(-4,0)$. It has gradient $\frac{1}{2}$. This gives $y = \frac{1}{2}(x+4)$, which we can rearrange to $2y = x+4$, as required.

737. This is a GP with ordinal formula $u_n = 5\cdot 4^{n-1}$. The boundary equation is
$$\begin{aligned}&10^9 = 5\cdot 4^{n-1}\\ \implies\ &2\times 10^8 = 4^{n-1}\\ \implies\ &n-1 = \log_4\left(2\times 10^8\right)\\ \implies\ &n = 14.7877...\end{aligned}$$
Since $n \in \mathbb{N}$, the first term whose value is over 1 billion is at $n = 15$. The value of the term is $u_{15} = 5\cdot 4^{14} = 1.34$ billion (3sf).

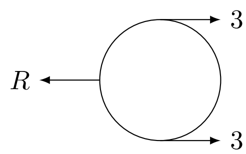
738. The second set contains all x values strictly within a distance 2 of 3 on the number line. This is the interval $(1,5)$. So, we have
$$[0,3] \cup (1,5).$$
The union of the two sets is $[0,5)$.

739. (a) The straight sections have total length 2 cm, and the curved sections have total length π cm. So, the length is $2+\pi$ cm.

i. The situation is symmetrical around the line of centres. The contact force between the canes can only act along this line of symmetry. Since this is perpendicular to the surfaces, it can only be a reaction force.

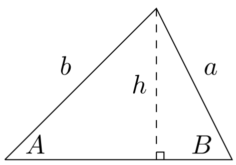
ii. If there were friction acting on the band, then the tension would differ along it. We are told that the tension is modelled as a constant 3 N throughout, so the contact has been modelled as smooth.

(b) Modelling the left-hand cane:



Equilibrium gives $R = 6$. There is no friction, so the contact force consists entirely of this reaction. The magnitude is 6 N.

740. Dropping a perpendicular in a general triangle ABC, we have



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The height h can be calculated in two different ways, using the two right-angled triangles formed by the perpendicular: $h = b\sin A$ and $h = a\sin B$. Equating these,

$$\begin{aligned} & b\sin A = a\sin B \\ \Longrightarrow\ & \frac{\sin A}{a} = \frac{\sin B}{b}. \end{aligned}$$

By symmetry, the same result holds for any other two side/angle pairs, which gives the full sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

— Nota Bene —

The version of the sine rule given above is the one used for finding angles. The version for finding lengths follows immediately, by reciprocating:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

741. The values of the parameter s at the endpoints of this line segment are $s = -1, 2$. Substituting these gives $(1, -7/3)$ and $(4, 5/3)$. By Pythagoras, the distance between these is $\sqrt{3^2 + 4^2} = 5$.

— Alternative Method —

As the parameter s changes by $\Delta s = 1$, the point generated changes by displacement $\mathbf{i} + 4/3\,\mathbf{j}$. By Pythagoras, the length of this vector is $5/3$. Hence, for $s \in [-1, 2]$, the distance is $3 \times 5/3 = 5$.

742. The graph has x intercepts at $x = -6$ and $x = 3$. So, using the factor theorem, the equation of the parabola takes the form $y = k(x+6)(x-3)$, for some constant k. Substituting $(0, 36)$ gives $k = -2$. Multiplying out to the form required, the equation is $y = -2x^2 - 6x + 36$.

743. There are 13 hearts and 13 diamonds. We require the first two cards to be hearts, and the second two diamonds. The cards are dealt with replacement, so, for each card, the probability of success is and remains at $\frac{1}{4}$. Hence, the probability is

$$\mathbb{P}(\text{HHDD}) = \left(\tfrac{1}{4}\right)^4 = \tfrac{1}{256}.$$

744. (a) This is $(-1, 1) \cap (-2, 2)$, which is $(-1, 1)$.

(b) This is exactly the same as (a). The variables x and y are defined and used within their sets, so $\{x \in \mathbb{R} : |x| < 2\}$ and $\{y \in \mathbb{R} : |y| < 2\}$ are exactly the same set $(-2, 2)$.

— Nota Bene —

The interchangeability of the variables x and y in the above is the same as that of s and t in the definite integrals

$$\int_a^b \mathrm{f}(s)\,ds \quad \text{and} \quad \int_a^b \mathrm{f}(t)\,dt,$$

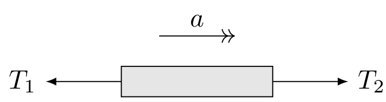
which are equal irrespective of the function f, and also that of i and j in the sums

$$\sum_{i=1}^{n} \mathrm{g}(i) \quad \text{and} \quad \sum_{j=1}^{n} \mathrm{g}(j),$$

which are equal irrespective of the function g. Such variables are used to encode instructions *internal* to an expression. Having done this job, they don't then feature. Hence, it makes no difference what symbol is used.

745. The quadratic has positive discriminant, and can be factorised as $(\cos x - 3)(\cos x + 2) = 0$. But the range of the cosine function is $[-1, 1]$, so neither of the factors can equal zero. Hence, the equation has no roots.

746. The force diagram is as follows:



$F = ma$ for the string is $T_2 - T_1 = ma$. If the string is modelled as light, then m is negligible, i.e. we set m equal to zero. Substituting this in gives $T_1 - T_2 = 0$. Hence, $T_1 = T_2$, as required.

747. False. The curves are unit circles centred on $(0, 0)$ and $(-1, -1)$. Since the distance between these two points is $\sqrt{2}$, which is less than 2, the circles must overlap and intersect twice.

748. Evaluation at $x = 1$ in the left-hand expression is not well defined, because the denominator $x^2 - x$ is equal to zero at $x = 1$. Both top and bottom have a factor of $(x - 1)$, but we cannot divide by it, since it is known to be zero.

In the right-hand expression, since we are only *approaching* 1, we can divide top and bottom by $(x-1)$, which is non-zero. After this, we can safely take the limit:

$$\begin{aligned} & \lim_{x\to 1} \frac{x^2 - 1}{x^2 - x} \\ =\ & \lim_{x\to 1} \frac{(x+1)(x-1)}{x(x-1)} \\ =\ & \lim_{x\to 1} \frac{x+1}{x} \\ =\ & 2. \end{aligned}$$

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749. The octahedron has eight symmetrical triangular faces. These produce eight symmetrical vertices, generating six square faces and twelve edges. The resulting polyhedron is a cube.

——— Nota Bene ———

It is an elegant result that not only is the dual of the octahedron the cube, but the dual of the cube is also the octahedron. Of the five Platonic solids, the cube (6) and the octahedron (8) are dual, the dodecahedron (12) and the icosahedron (20) are dual, and the tetrahedron (4) is self-dual.

750. Solving for intersections,

$$\begin{aligned} & x^2 = 2px - p^2 \\ \Longrightarrow\ & x^2 - 2px + p^2 = 0. \end{aligned}$$

This has discriminant $\Delta = 4p^2 - 4 \cdot p^2 = 0$, for any p. Hence, since they have only one intersection, the line is tangent to the curve.

——— Alternative Method ———

The point (p, p^2) lies on both the curve $y = x^2$ and the line $y = 2px - p^2$. Furthermore, the gradient of $y = x^2$ at this point is

$$\left.\frac{dy}{dx}\right|_{x=p} = 2p,$$

which is the gradient of the line. Hence, the line is tangent to the curve.

751. A nanocentury is $10^{-9} \cdot 100 \cdot 365 \cdot 24 \cdot 60 \cdot 60$ seconds. This is $3.1536 \approx \pi$, as required.

752. (a) $16p^2q^2 - 1 = 0$, which can be factorised as $(4pq - 1)(4pq + 1) = 0$.

(b) From part (a), we have $pq = \pm\frac{1}{4}$. Since p and q are probabilities, $p, q \geq 0$, so the negative root is not possible. Hence, we have simultaneous equations $pq = \frac{1}{4}$ and $p + q = 1$. Solving these gives $p = q = \frac{1}{2}$.

753. The squared factor can be zero, at $x = 0$, which means that the minimum value of the squared term is zero. Hence, the range is $[4, \infty)$.

754. (a) When $t = 0$, $\mathbf{r} = \mathbf{b}$, and when $t = 1$, $\mathbf{r} = \mathbf{a}$. Hence, the line L passes through A and B.

(b) Having set up a vector equation in this way, with $t = 0$ at B and $t = 1$ at A, the parameter t acts like a scale running linearly between A and B. If AB is divided $1 : 3$, then $t = \frac{3}{4}$. So, the position vector is $\mathbf{r} = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$.

755. If a function f has a linear and non-constant first derivative, then $\mathrm{f}'(x) = ax + b$, for some $a \neq 0$. Differentiating both sides of this gives

$$\mathrm{f}''(x) = a.$$

This is a linear function, but it is constant. Hence, no function has linear first and second derivatives that are non-constant. □

756. Multiplying out, the equation is

$$abx^2 + (a^2 + b^2)x + ab = 0.$$

Since $a, b \neq 0$, this is quadratic. It has precisely one real root, so $\Delta = 0$:

$$\begin{aligned} & \left(a^2 + b^2\right)^2 - 4a^2b^2 = 0 \\ \Longrightarrow\ & a^4 + 2a^2b^2 + b^4 - 4a^2b^2 = 0 \\ \Longrightarrow\ & a^4 - 2a^2b^2 + b^4 = 0 \\ \Longrightarrow\ & \left(a^2 - b^2\right)^2 = 0 \\ \Longrightarrow\ & a^2 = b^2, \text{ as required.} \end{aligned}$$

——— Alternative Method ———

The equation is quadratic. It has precisely one real root, so it must have a squared factor. Hence, $(ax + b)$ and $(bx + a)$ must be scalar multiples of one another. Equating the scale factors between the coefficient of x and the constant term, we get $a/b = b/a$. This gives $a^2 = b^2$, as required.

757. (a) Since $(p - q)$ and $(q - p)$ are negatives, this simplifies to -1.

(b) Using the same fact,

$$\begin{aligned} \frac{q^2 - p^2}{p - q} &\equiv \frac{(q + p)(q - p)}{p - q} \\ &\equiv -p - q. \end{aligned}$$

758. Using the cosine rule to find the angle opposite the 14 cm edge,

$$\cos\theta = \frac{9^2 + 13^2 - 14^2}{2 \cdot 9 \cdot 13} = \tfrac{3}{13}.$$

Converting from cosine to sine, we use the first Pythagorean trig identity. Since the triangle is acute, we take the positive root, giving

$$\sin\theta = \sqrt{1 - \left(\tfrac{3}{13}\right)^2} = \tfrac{4\sqrt{10}}{13}.$$

Hence, the area is

$$A_\triangle = \tfrac{1}{2} \cdot 9 \cdot 13 \cdot \tfrac{4\sqrt{10}}{13} = 18\sqrt{10}.$$

——— Alternative Method ———

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FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

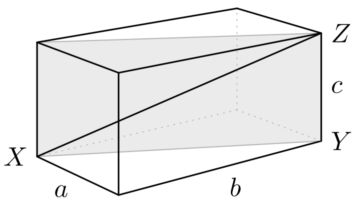
Heron's formula gives the area of a triangle in terms of the semiperimeter s as

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Here, the semiperimeter is $s = \frac{1}{2}(9+13+14) = 18$. So, the area is

$$\begin{aligned} A &= \sqrt{18(18-9)(18-13)(18-14)} \\ &= \sqrt{3240} \\ &= 18\sqrt{10}, \text{ as required.} \end{aligned}$$

759. In the diagram below, we need to calculate $|XZ|$ using 2D Pythagoras:



By 2D Pythagoras, $|XY|^2 = a^2 + b^2$. Then, using 2D Pythagoras again in triangle XYZ,

$$\begin{aligned} |XZ|^2 &= |XY|^2 + c^2 \\ &= a^2 + b^2 + c^2. \end{aligned}$$

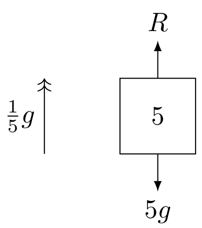
This proves Pythagoras's theorem in 3D.

760. Writing the sum longhand,

$$\begin{aligned} &\sum_{i=1}^{3} \cos \frac{\pi \text{ rad}}{i} \\ =\ &\cos\tfrac{\pi}{1} + \cos\tfrac{\pi}{2} + \cos\tfrac{\pi}{3} \\ =\ &-1 + 0 + \tfrac{1}{2} \\ =\ &-\tfrac{1}{2}. \end{aligned}$$

761. (a) The force exerted by the pirate's feet on the lift floor is the same as the reaction force exerted by the lift floor on the pirate's feet. Treating the pirate/parrot as one object of mass 85 kg, this force is $85g$ N, since the acceleration is zero.

(b) The force diagram for the parrot is



By NIII, the force exerted by the parrot's feet on the pirate's shoulder is the same as the force exerted by the pirate's shoulder on the parrot's feet. This is R in the diagram above. NII gives $R - 5g = 5 \cdot \frac{1}{5}g$, so $R = 6g$. The parrot's feet exert $6g$ N downwards on the pirate's shoulder.

762. The number n of roots of the equation $\mathrm{f}(x) = a$, listing in order from the uppermost element of the codomain C, is $3, 0, 0, 0, 2$. So, $n \in \{0, 2, 3\}$.

763. This is a quadratic in $\sin x$. Factorising,

$$\begin{aligned} &\sin x(\sin x + 1) = 0 \\ \Longrightarrow\ &\sin x = 0, -1. \end{aligned}$$

The former yields $x = 0°, 180°$, the latter $x = 270°$. So, the solution is $x \in \{0, 180, 270°\}$.

764. The equation for intersections is

$$\begin{aligned} &x^2 + kx + k = x + k \\ \Longrightarrow\ &x^2 + (k-1)x = 0. \end{aligned}$$

Since this has no constant term, $x = 0$ is always a root, so the line and the curve have at least one intersection.

— Alternative Method —

The equation for intersections is $x^2 + (k-1)x = 0$, as above. The discriminant of this quadratic is $\Delta = (k-1)^2$. Since this is always non-negative, the graphs must have at least one intersection.

765. To calculate the *expected* change, we can assume that the selection of twenty is representative of the hundred. So, 20% of the sample is scaled by 0.75 and the remaining 80% is not scaled (scaled by factor 1). The overall scale factor associated with this decrease is then

$$0.2 \times 0.75 + 0.8 \times 1 = 0.95.$$

So, the expected percentage reduction is 5%.

766. If the two right angles were next to one another, then, by co-interior angles, two of the edges would be parallel. We are told that this is not the case. So, the right angles must be opposite one another.

Right angles at opposite vertices add up to 180°: they are complementary. Hence, the quadrilateral is cyclic, according to the circle theorem of the same name. So, there is a circle passing through all four vertices. $\square$

767. If α is a fixed point of f, then

$$\mathrm{f}(\alpha) = \alpha.$$

We can apply the inverse f^{-1} to both sides. This undoes f on the LHS, giving

$$\alpha = \mathrm{f}^{-1}(\alpha).$$

Hence, α is a fixed point of the inverse f^{-1}. QED.

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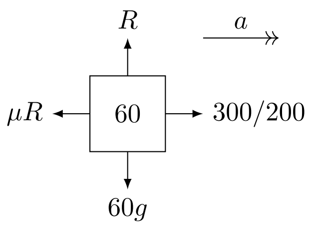
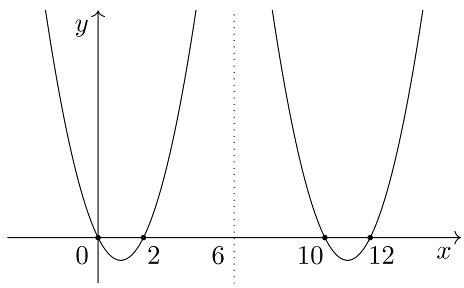
768. The force diagram for both parts, assuming that friction is limiting at $F_{\text{max}} = \mu R$, is



(a) Vertical equilibrium gives $R - 60g = 0$. Hence, limiting friction is $F_{\text{max}} = 24g$. Horizontally, NII is $300 - 24g = 60a$, which gives $a = 1.08$ ms^{-2}. This is positive, so the assumption of limiting friction was correct.

(b) Again, $F_{\text{max}} = 24g$. Then $200 - 24g = 60a$ gives $a = -0.587$ ms^{-2}. But friction acting to oppose motion cannot generate such a negative acceleration. So, in this case, the assumption of limiting friction was incorrect. Friction is in fact less, at exactly 200 N, and the acceleration is 0 ms^{-2}.

769. The monic parabola $y = x^2 - 2x = x(x-2)$ has x intercepts at $x = 0$ and $x = 2$. Reflecting this in $x = 6$, the new parabola must be monic and have roots at $x = 10$ and $x = 12$.



Using the factor theorem, the second parabola is $y = (x-10)(x-12)$ or $y = x^2 - 22x + 120$.

770. Since opposite faces of a die add up to seven, the vertical faces of each die total to $2 \times 7 = 14$. The maximum is then attained when the uppermost die has six showing on its top face, giving a maximum total of $3 \times 14 + 6 = 48$.

771. (a) These events sum to 1. So $x + 3x + \frac{1}{5} + 4x = 1$, which gives $x = \frac{1}{10}$.

(b) If A and B are independent, then so are A' and B'. The definition of independence for A' and B' is

$$\mathbb{P}(A' \cap B') = \mathbb{P}(A') \times \mathbb{P}(B').$$

Substituting in values,

$$\begin{aligned}\tfrac{4}{10} &= \mathbb{P}(A') \times \tfrac{5}{10} \\ \Longrightarrow \mathbb{P}(A') &= \tfrac{4}{5}.\end{aligned}$$

772. In the following, as throughout mechanics, the phrase "A is negligible" means "A is close enough to zero that, compared with other quantities, its effects can be neglected by setting it to zero."

(a) An object, usually a string or rope or similar, is called *inextensible* if it undergoes negligible extension, i.e. if it doesn't stretch.

(b) An object is called *uniform* if it has negligible deviation from constant density, i.e. it is made of the same material throughout.

(c) An object, often a rod or beam or suchlike, is called *rigid* if, when forces are applied to it, it undergoes negligible deformation of any kind, i.e. it has a fixed shape.

773. We know that $2 = 16^{\frac{1}{4}}$. Hence, using an index law to switch the order in which the indices are applied:

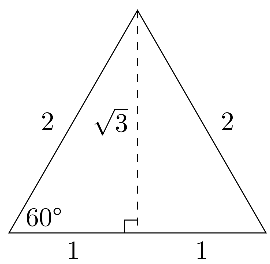
$$\begin{aligned}2^t &\equiv \left(16^{\frac{1}{4}}\right)^t \\ &\equiv \left(16^t\right)^{\frac{1}{4}}.\end{aligned}$$

774. (a) We eliminate x as below:

$$\begin{aligned}3A - B &: -13y + 13z - 13 = 0 \\ 2B - C &: -y - 6z + 41 = 0.\end{aligned}$$

(b) Solving the equations in part (a), we get $y = 5$ and $z = -6$. In turn, this gives $x = -2$.

775. In an equilateral triangle of side length 2, drop a perpendicular to split the triangle into two right-angled triangles. By Pythagoras, these have sides $(1, \sqrt{3}, 2)$.



Hence, $\sin 60° = \frac{\sqrt{3}}{2}$, as required.

776. The equations are two straight lines. For these to have infinite intersections, they must be identical. Rearranging, this gives $k = \frac{7}{3}$.

777. Since the y intercepts c_n form an AP, the distance in the y direction between successive pairs of lines is constant. Call it d_1. The distance between the lines is then given by d_2 in the following diagram:

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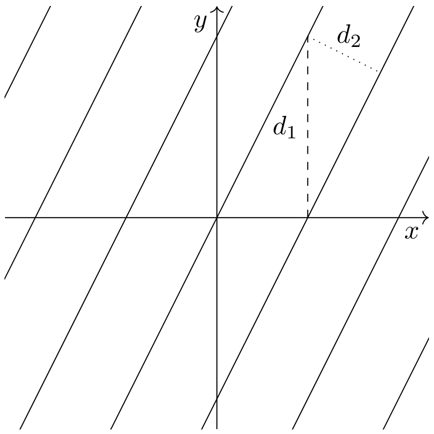


Since lines are parallel, the scale factor between d_1 and d_2 is constant, whichever pair of adjacent lines are chosen. So, since the distances d_1 are constant, the distances d_2 are also constant.

778. (a) i. The cable is light, so these two tensions are effectively a Newton III pair. Hence, they are equal by definition.

ii. Since the masses of the spacecraft and the astronaut are different, the magnitudes of the accelerations, by NII, will be different.

(b) For the five seconds of tension, $a_1 = 0.05$ and $a_2 = 0.75$. So, the relative acceleration is $a = 0.8$. After these five seconds, the distance has been reduced by $s = \frac{1}{2} \cdot 0.8 \cdot 5^2 = 10$ metres, and the relative speed is now $v = 0.8 \cdot 5 = 4$ ms^{-1}. The remaining 40 metres is covered in 10 seconds, giving 15 seconds overall.

779. Let $\mathrm{f}(x) = ax + b$, where a and b are constants. Carrying out the integral

$$\begin{aligned} &\int_{-2}^{2} ax + b \, dx = 1 \\ \implies &\left[\tfrac{1}{2}ax^2 + bx\right]_{-2}^{2} = 1 \\ \implies &(2a + 2b) - (2a - 2b) = 1 \\ \implies &b = \tfrac{1}{4}. \end{aligned}$$

Hence, irrespective of the value of a, $\mathrm{f}(0) = \frac{1}{4}$.

780. The mean and median, as two measures of central tendency, are negated. But the three measures of spread, all of which are non-negative by definition, are not.

(a) True.
(b) False.
(c) True.
(d) False.
(e) False.

781. We write the proportionality with constant k, and integrate with respect to z:

$$\frac{dy}{dz} = k\frac{dx}{dz} \implies y = kx + c.$$

(a) It is not necessarily true (though possible) that y and x are directly proportional.

(b) It is necessarily true that y and x are linearly related.

782. The quantity $x^4 + y^4$ increases with distance from the origin. Hence, we test the quantity $x^4 + y^4$, comparing it to 1. Carrying out the evaluation,

$$x^4 + y^4 \Big|_{(0.8, 0.9)} = 1.0657 > 1.$$

So, the point lies outside the curve.

Nota Bene

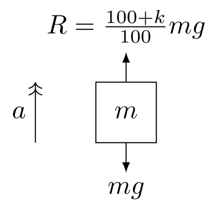
The fact that $x^4 + y^4 = 1$ is a closed curve (a loop) is the same fact as $x^4 + y^4$ increasing with distance. Both facts stem from the index being even, such that x^4 and y^4 are non-negative.

In contradistinction to the above, the quantity $x^3 + y^3$ does not increase with distance from the origin (it is zero at all points on the line $y = -x$), and the curve $x^3 + y^3 = 1$ is duly open: it stretches to infinity asymptotically along $y = -x$.

783. The result is correct, but a piece of justification has been left out, which is necessary if the solution is to be *determined* rather than merely *found*. When dividing by an expression, the possibility that that expression could be zero should be considered. In this case, a better argument would read

"Since 2^{2x} is necessarily positive, we can divide both sides by 2^{2x}, which gives $2^x = 5$. Hence, $x = \log_2 5$."

784. Scales measure contact force, specifically reaction force, converting from a reaction force R to a mass m via $R = mg$. If mass is overestimated by $k\%$, then the reaction force R is $k\%$ greater than mg. This is depicted below:



The equation of motion is

$$\begin{aligned} \frac{100 + k}{100}mg - mg &= ma \\ \implies a &= \frac{kg}{100} \end{aligned}$$

The acceleration upwards is $k\%$ of g, or $\frac{kg}{100}$ ms^{-2}.

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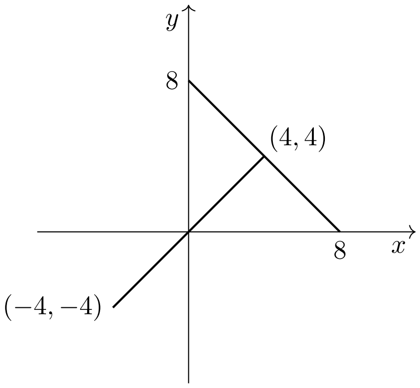
785. The endpoints of the segments are at

	x	y
$s = 0$	0	8
$s = 8$	8	0
$t = -4$	-4	-4
$t = 4$	4	4

Joining the dots, the line segments are:



This forms a capital T.

786. Evaluating the integral directly using a calculator definite integration facility

$$\int_0^{\frac{\pi}{6}} \cos\theta \, d\theta = \tfrac{1}{2}.$$

Evaluating again on a calculator, this time using the small-angle approximation, gives

$$\int_0^{\frac{\pi}{6}} 1 - \tfrac{1}{2}\theta^2 \, d\theta = 0.499674...$$

So, the percentage error is

$$\frac{0.499674 - 0.5}{0.5} = -0.065\% \text{ (3sf)}.$$

787. There is a common factor of $(x+1)$ on the LHS, which corresponds to a root at $x = -1$. Dividing through by this factor,

$$\begin{aligned} & 1 + \frac{x+1}{x^2+1} = 0 \\ \Longrightarrow\ & x^2 + 1 + x + 1 = 0 \\ \Longrightarrow\ & x^2 + x + 2 = 0. \end{aligned}$$

The discriminant of this quadratic is $\Delta = -7 < 0$, so it has no real roots. Hence, the solution of the original equation is $x = -1$.

788. Since neither interval is a subset of the other, there is no implication statement that joins these two. All four truth combinations are possible, as shown by the counterexamples below:

x	0.5	1.5	2.5	3.5
$x \in [0, 2]$	T	T	F	F
$x \in [1, 3]$	F	T	T	F

789. Since $(2x - \beta)$ divides exactly into the quadratic, $x = \beta/2$ must be a root, by the factor theorem. Hence,

$$\begin{aligned} & 2\beta^2 + \frac{\beta}{2} + \beta = 0 \\ \Longrightarrow\ & 3\beta^2 + \beta = 0 \\ \Longrightarrow\ & \beta = 0, -3. \end{aligned}$$

790. This is a positive cubic with a double root at $x = 2$ and a single root at $x = -3$. Hence, it has the form

$$y = k(x-2)^2(x+3).$$

The y intercept tells us that $k = 2$. Multiplying out gives $y = 2x^3 - 2x^2 - 16x + 24$.

791. Assume, for a contradiction, that there are real numbers $a, b \neq 0$ such that $a\mathbf{p} + b\mathbf{q} = 0$. Since $a \neq 0$, we can rearrange to

$$\mathbf{p} = -\tfrac{b}{a}\mathbf{q}.$$

But $-\frac{b}{a}$ is a scalar, which means that $\mathbf{p}$ and $\mathbf{q}$ must be parallel. This is a contradiction. Hence, there are no non-zero real numbers a and b such that $a\mathbf{p} + b\mathbf{q} = 0$. $\square$

792. Integrating twice with respect to x gives

$$\begin{aligned} & \frac{d^2y}{dx^2} = 2a \\ \Longrightarrow\ & \frac{dy}{dx} = 2ax + b \\ \Longrightarrow\ & y = ax^2 + bx + c. \end{aligned}$$

The general solution is all quadratic curves of the form $y = ax^2 + bx + c$.

793. The second equation may be written as $v = \frac{2s}{t} - u$. Substituting this into the first equation gives

$$\begin{aligned} & \left(\frac{2s}{t} - u\right)^2 = u^2 + 2as \\ \Longrightarrow\ & \frac{4s^2}{t^2} - 4sut + u^2 = u^2 + 2as \\ \Longrightarrow\ & \frac{4s^2}{t^2} - 4sut = 2as. \end{aligned}$$

Multiplying by $\frac{4t^2}{s}$,

$$\begin{aligned} & s - ut = \tfrac{1}{2}at^2 \\ \Longrightarrow\ & s = ut + \tfrac{1}{2}at^2. \end{aligned}$$

— Nota Bene —

We have divided by t and s in the above, so should consider division by zero.

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- If $t = 0$, then the result is trivial. There is no period of constant acceleration, and the displacement s must be zero, in line with the formula.
- If $s = 0$ and $t \neq 0$, then the result is not trivial. The first equation becomes $v^2 = u^2$. The second equation becomes $\frac{1}{2}(u + v)t = 0$, which, since $t \neq 0$, is $u = -v$.

  Each equation is a link between u and v, and neither provides information about a or t. So, in the case $s = 0$, the formula $s = ut + \frac{1}{2}at^2$ cannot be derived from these two.

794. Substituting $x = 0$ gives $A = \pm 1$, and substituting $x = 1$ gives $B = \pm 4$. We now equate coefficients of x^4. On the LHS, the coefficient is 1. On the RHS, it is $(A+B)^2$. There is no combination of $A = \pm 1$ and $B = \pm 4$ which generates $(A + B)^2 = 1$. So, there are no values of the constants A and B which make the identity hold.

——— Alternative Method ———

Expanding binomially, the LHS is

$$(x + 1)^4 \equiv x^4 + 4x^3 + 6x^2 + 4x + 1.$$

The RHS is

$$\begin{aligned} &\Big(A(x-1)^2 + Bx^2\Big)^2 \\ = &\Big((A + B)x^2 - 2Ax + A\Big)^2. \end{aligned}$$

Equating constant terms, $A = 1$. Equating the coefficients of x^4, $A + B = 1$, so $B = 0$. But this gives the RHS as

$$(x^2 - 2x + 1)^2 \equiv x^4 - 4x^3 + 6x^2 - 4x + 1.$$

The terms in x^3 and x do not match, so there are no values of the constants A and B which make the identity hold.

795. For the tests to have equal weight, they must each be scaled to be out of 50 marks. This requires scale factors of 5/4 and 5/6 respectively. Hence, the formula is $X = \frac{5}{4}A + \frac{5}{6}B$.

796. (a) i. We can factorise $1 - 2r + r^2$ to $(1 - r)^2$. Being a square, this cannot be negative.

   ii. The GP is positive, meaning that all of its terms are positive. Hence, $a > 0$. So, when multiplying the first result by a, the second result is also non-negative.

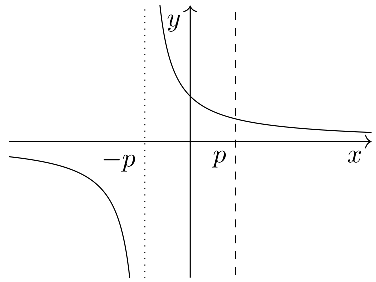
   (b) Using the second result of part (a),

$$\begin{aligned} & a - 2ar + ar^2 \geq 0 \\ \Longrightarrow\ & a + ar^2 \geq 2ar \\ \Longrightarrow\ & \frac{a + ar^2}{2} \geq ar. \end{aligned}$$

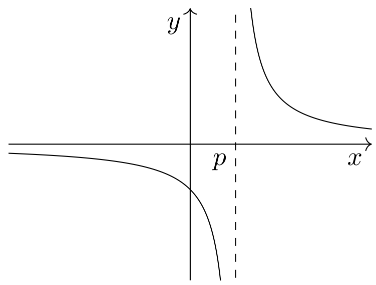
The LHS is the mean of u_1 and u_3, and the RHS is u_2. Hence, u_2 cannot be bigger than the mean of u_1 and u_3. □

797. These are reciprocal graphs: translated versions of $y = \frac{1}{x}$. And the line $x = p$ is parallel to the y axis. Such a line will intercept a reciprocal graph for any x value other than that of the vertical asymptote.

   (a) The line $x = p$ intersects $y = \dfrac{1}{x + p}$:



   (b) The line $x = p$ does not intersect $y = \dfrac{1}{x - p}$:



798. The three odd squares are odd numbers. So, they can be written $2p + 1$, $2q + 1$ and $2r + 1$, where $p, q, r \in \mathbb{Z}^+$. The sum is $2(p + q + r) + 3$. This is 3 greater than an even number, so is odd. □

——— Alternative Method ———

For $a, b, c \in \mathbb{Z}$, the sum of three odd squares is

$$(2a + 1)^2 + (2b + 1)^2 + (2c + 1)^2.$$

This simplifies to $4(a^2 + a + b^2 + b + c^2 + c) + 3$. The first term is a multiple of 4, so is even. Adding 3 produces an odd number. So, the sum of three odd squares is odd. □

799. Attaining different suits is more probable without replacement. Removal of the first card from the possibility space means that more cards of other suits (i.e. successful outcomes) remain.

800. (a) This does not hold. Consider $a = b \neq c$, with $\mathrm{f}(c) \neq 0$. This is a counterexample.

   (b) This holds. If $a = c$, then $\mathrm{f}(c) = 0$. And, since c is a root of $\mathrm{f}(x) = \mathrm{g}(x)$, $\mathrm{f}(c) = \mathrm{g}(c)$. So, $\mathrm{g}(c) = 0$, proving the result.

——— End of 8th Hundred ———

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP